EFFECTS OF SYMMETRY ON PARADOXICAL COST DEGRADATION IN A NASH NON-COOPERATIVE NETWORK SYSTEM

Hisao Kameda ∗,2 Makoto Ohta ∗,3 Yoshihisa Hosokawa ∗,4

Abstract: This paper interprets globalization as increased transportation capacity in a system, and investigates its effects when the system is regarded as a non-cooperative game. It is shown that there exist cases where increased capacity degrades the benefits to all the players. Further, complete symmetry may be harmful in the sense that our measure of degradation becomes worst, where our measure is the minimum of the ratios of the cost degradation for all the players. The system considered consists of two locations each of which has a facility that can produce a commodity, a local fixed demand for the commodity, and a player.

Keywords: Distributed decision, Braess paradox, prisoners’ dilemma, Nash equilibrium, cost optimization, Pareto inefficiency.

1. INTRODUCTION

Adding transportation means among separated locations introduces a chance of sharing the facilities among separated locations. It would be anticipated that the benefits of, at least, some locations would be increased by adding transportation capacity between the locations. This is not always the case, however, as exemplified in the Braess paradox in transportation and communication networks (Calvert et al., 1997; Cohen and Kelly, 1990; Cohen and Jeffries, 1997; Korilis et al., 1995; Korilis et al., 1999), the Braess-like paradox in distributed computer systems (Kameda et al., 2000), or, more generally, the payoff matrix of the well-known prisoners’ dilemma, etc. That is, there exist cases where the costs of all locations degrade by such addition of capacity. The famous Braess paradox shows that adding capacity to a system may sometimes degrade the performance of all users in the Wardrop equilibrium (Calvert et al., 1997; Cohen and Kelly, 1990; Cohen and Jeffries, 1997). Some examples where a paradox similar to Braess’s appear in a Nash equilibrium have been found (Calvert et al., 1997; Cohen and Kelly, 1990; Cohen and Jeffries, 1997).

Furthermore, numerical investigation (Hosokawa et al., 2000), an extension of a result (Kameda et al., 2000), suggested the conjecture that the paradoxical cost degradation due to increased capacity is worst in complete symmetry, i.e., the minimum among the ratios of the degradation for all the players is largest when the values of parameters describing the players (except their decisions) are identical.

It is desirable to gain more insight into the conditions under which such paradoxical cost degradation occurs and to examine the above conjecture. Although the problem may occur very generally, it would be difficult to analyze a model that covers all possible situations. The specialized model treated here is analytically tractable, yet already admits interesting results. These results suggest general characteristics that may be applicable to more general situations.

The present model consists of two locations, each of which has one facility. Each facility can produce a commodity of single kind, common to both locations.

2 Institute of Information Sciences and Electronics, University of Tsukuba, Tsukuba Science City, Ibaraki 305-8573, Japan. E-mail: kameda@is.tsukuba.ac.jp
3 Institute of Policy and Planning, E-mail: ohta@sk.tsukuba.ac.jp
4 Graduate School of Systems and Information Engineering, E-mail: hosokawa@osdp.is.tsukuba.ac.jp
At each location, a fixed demand of the commodity is created. One decision maker exists for each location, and optimizes the cost of producing the amount of the commodity demanded in the location non-cooperatively. The optimized situation where neither location can receive further benefit by changing its decision is called Nash equilibrium.

It is counter-intuitive that addition of transportation capacity to such a system may induce cost degradation for all decision makers. The present paper characterizes the cases where paradoxical cost degradation does and does not occur. A measure of paradoxical cost degradation is defined as the minimum of the ratios of the cost degradation for all the players (Kameda et al., 2000). It is shown that, in some situations, the measure is worst in complete symmetry, i.e., when the values of parameters describing the two facilities and the demands are identical.

The description of the model investigated is given in Section 2, and Section 3 presents the characterization of the paradoxical cases (in Subsection 3.2), and shows that our measure of the paradoxical cost degradation is worst in some cases of completely symmetry among the locations (in Subsection 3.3). Section 4 concludes this paper.

2. THE MODEL AND ASSUMPTIONS

The system considered here has two distinct locations and a transportation means that connects both locations. Locations are numbered 1 and 2. In both locations, a common single kind of commodity is demanded. Assume that, at locations 1 and 2, respectively, the quantities $\xi$ and $\eta$ of the commodity are demanded. There is one production facility at each location. Facilities are numbered 1 and 2 according to their locations. Each location has one decision maker, also numbered 1 and 2.

Let $x$ and $y$ denote, respectively, the amounts of the commodity that decision makers 1 and 2 order from facilities 2 and 1. Thus, $\xi - x$ and $\eta - y$ denote, respectively, the amounts of production that decision makers 1 and 2 demand from their own facilities. Define the vector $x$ by $x = (x, y)$. Thus

$$0 \leq x \leq \xi \quad \text{and} \quad 0 \leq y \leq \eta. \quad (1)$$

Denote by $C$ the set of $x$ that satisfy the above constraints (1).

$x$ and $y$, respectively, are the variables controlled by decision makers 1 and 2.

Consider a simple case where the costs that facilities 1 and 2, respectively, require in producing a unit amount of the commodity are $a + bx$ and $c + dy$ ($a \geq 0, b \geq 0, c \geq 0$) when they are required to produce the amounts $u$ and $v$ of the commodity. Thus, if each decision maker decides to produce its entire demand in its own facility, the total costs $C_{10}$ and $C_{20}$, respectively, in producing the amount $\xi$ and $\eta$ of the commodity in the facilities 1 and 2 are

$$C_{10} = \xi(a + bx) \quad \text{and} \quad C_{20} = \eta(c + dy). \quad (2)$$

Assume that, if a decision maker orders from the facility at the other location, he must compensate the other location for the production cost, and an extra transportation cost of $t$ per unit of the commodity is added. Thus, if decision makers 1 and 2, respectively, decide to order amounts $x$ and $y$ to be produced by other facility, the total costs $C_1(x, y)$ and $C_2(x, y)$ for the decision makers 1 and 2 are the following:

$$C_1(x, y) = (\xi - x)[a + b(\xi - x + y)] + [c + d(\eta - y + x)] + xt, \quad (3)$$

$$C_2(x, y) = (\eta - y)[c + d(\eta - y + x)] + [a + b(\xi - x + y)] + yt. \quad (4)$$

Therefore, the optimal decisions by decision makers 1 and 2, respectively, are expressed as follows:

$$\min_x C_1(x, y) \quad \text{such that} \quad 0 \leq x \leq \xi \quad \text{given} \quad y, \quad (5)$$

$$\min_y C_2(x, y) \quad \text{such that} \quad 0 \leq y \leq \eta \quad \text{given} \quad x. \quad (6)$$

Denote by $\hat{x} = (\hat{x}, \hat{y})$ such values of $x$ and $y$ that satisfy

$$C_1(\hat{x}, \hat{y}) = \min_x C_1(x, \hat{y}) \quad \text{such that} \quad 0 \leq x \leq \xi \quad \text{given} \quad \hat{y}, \quad (7)$$

$$C_2(\hat{x}, \hat{y}) = \min_y C_2(\hat{x}, y) \quad \text{such that} \quad 0 \leq y \leq \eta \quad \text{given} \quad \hat{x}. \quad (8)$$

If the system has a solution $\hat{x}$ of $x$ that satisfies (7) and (8) at the same time, it is a Nash equilibrium.

In contrast, suppose the system has only one decision maker who optimizes the entire cost $C_1 + C_2$ of the system. The solution $\hat{x}$ of such perfectly coordinated optimization satisfies:

$$\min_{x, y} C_1(x, y) + C_2(x, y) \quad \text{such that} \quad 0 \leq x \leq \xi, 0 \leq y \leq \eta. \quad (9)$$

3. THE RESULTS

3.1 The case where a single decision maker optimizes the overall cost of the system

It can be easily seen that at least one of $\hat{x}$ and $\hat{y}$ must be zero in the overall optimal solution to the problem (9). For example, if it were the case that $0 < \hat{x} \leq \hat{y}$, then it would follow that $C_1(\hat{x}, \hat{y}) + C_2(\hat{x}, \hat{y}) > C_1(0, \hat{y} - \hat{x}) + C_2(0, \hat{y} - \hat{x})$, which is a contradiction. Thus, clearly, it is impossible that $C_1(\hat{x}, \hat{y}) > C_{10}$ and $C_2(\hat{x}, \hat{y}) > C_{20}$ hold at the same time, i.e., paradoxical performance degradation due to adding capacity should never occur in the overall optimization scheme.

3.2 The cases where paradoxes do not occur

Proposition 1. The system has a unique Nash equilibrium $\hat{x}$ as follows:
Each case of the solution reflects the corresponding condition of the following:

\[ 0 < C_{11}(0,0) & 0 < C_{22}(0,0) \quad [1(1)] \\
0 < C_{11}(0,\bar{y}) & \& C_{22}(0,0) \leq C_{22}(0,\bar{y}) = 0 \leq C_{22}(0,\bar{y}) \quad [2(2)]
\]
\[ 0 < C_{11}(0,\bar{y}) \& C_{22}(0,\bar{y}) < 0 \quad [3(1)] \\
C_{11}(0,\bar{y}) \leq C_{11}(0,\bar{y}) = 0 \leq C_{11}(\hat{x},\bar{\eta}) \quad [2(1)] \\
\& C_{22}(0,\bar{y}) \leq C_{22}(0,\bar{y}) = 0 \leq C_{22}(\hat{x},\bar{\eta}) \quad [2(2)] \\
C_{11}(0,\bar{y}) < 0 & 0 < C_{22}(\hat{x},\bar{\eta}) \quad [2(3)] \\
C_{11}(\hat{x},\bar{\eta}) < 0 & C_{22}(\hat{x},\bar{\eta}) < 0 \quad [3(1)] \\
C_{11}(\hat{x},\bar{\eta}) < 0 & C_{22}(\hat{x},\bar{\eta}) < 0 \quad [3(3)] \\
\]

Relation (10) is easily obtained from the above. Q.E.D.

Now consider the degree to which the benefit of each decision maker is improved or degraded due to adding the transportation means between the two locations. Define \( C_i \uparrow C_i(\bar{x}), i = 1, 2 \). Let \( k_i \) denote the ratio \( C_i/C_0 \) for \( i = 1, 2 \). If \( k_1 < 1 \) and \( k_2 < 1 \), both decision makers receive increased benefits. If \( k_1 \geq 1 \) and \( k_2 < 1 \) for some \( i, j \) \( (i \neq j) \), the situation is not paradoxical yet, and only from this it cannot be said that the situation is Pareto inefficient. If both \( k_1 > 1 \), \( i = 1, 2 \), the situation is paradoxical like the Braess paradox and the prisoners' dilemma, and is clearly Pareto inefficient. Call this situation paradoxical cost degradation.

The following property is obtained:

**Proposition 2.** In the case where \( b + d \leq 0 \), no paradoxical cost degradation occurs.

**[Proof]** Consider \( (0, \eta) \). For this to be a solution, both of the following must hold simultaneously: \( C_1(0, \eta) \leq C_1(0,0) = C_1(0,0) \leq C_2(0,\bar{\eta}) \). From the former, it follows that \( k_1 \leq 1 \).

For \( (0, \eta) \), both of the following must hold simultaneously: \( C_2(0, \eta) \leq C_2(0,0) = C_2(0,0) \leq C_1(\hat{x},\bar{\eta}) \). From the former, it follows that \( k_1 \leq 1 \).

In order that \( (\xi, \eta) \) be a solution, both of the following must hold simultaneously:
\[ C_1(0, \eta) \geq C_1(\hat{x},\bar{\eta}) \quad (\xi, \eta) \geq C_2(\xi, \eta) \]

The former is identical with \( 0 < c + d\xi + t - (a + b\eta) \leq b\xi \), and the latter \( a + b\eta + t - (c + d\xi) \leq d\eta \). Then it holds that \( 2t - (b\xi + d\eta) \leq 0 \). Therefore,
\[ (\hat{C}_1 - C_1(0,0) + (C_2 - C_20)/\eta) = [c + d\xi + t - (a + b\eta)] + [a + b\eta + t - (c + d\eta)] = (b \cdot d)(\eta + 2t) = (b \cdot d)(\xi + \eta) = 2t - (b\xi + d\eta) = 2t \]
\[ \leq (b \cdot d)(\xi + \eta) = 2t - (b\xi + d\eta) \]
\[ \]

It thus holds that \( \hat{C}_1 - C_10 < 0 \) or \( \hat{C}_2 - C_20 < 0 \), and \( k_1 < 1 \) or \( k_2 < 1 \). Q.E.D.

**Proposition 3.** In the case where \( b + d > 0 \), no paradoxical degradation occurs when the solution has either \( \hat{x} = 0 \) or \( \hat{y} = 0 \) and when it is \( (\xi, \eta) \).
[Proof] Consider the case where $\xi = 0$, i.e., cases (1-1), (1-2), and (3-1). It is trivial for case (1-1) where $C_1(0,0) = C_{10}$ and $C_2(0,0) = C_{20}$, and it holds that $k_1 = k_2 = 1$.

For cases (1-2) and (1-3), it is easily seen that $C_2(0,0) < C_2(0,0) = C_{20}$ and $k_2 < 1$.

For cases (2-1) and (3-1) ($\bar{y} = 0$), similarly $k_1 < 1$.

For cases (3-2), from (10), the following holds:

$$a - c + 2b\xi - b\eta + t > (2\xi - \eta)(b + d),$$

$$c - a + 2b\xi - b\eta - t > (2\eta - \xi)(b + d).$$

From the above two relations, it follows that

$$d\xi + b\eta + 2t < 0.$$  

Therefore,

$$(\bar{C}_1 - C_{10})/\xi + (\bar{C}_2 - C_{20})/\eta = [c + d\xi + t - (a + b\xi)] + [a + b\eta - c + d\eta] = (b - d)\eta - \xi + 2t = 2d\xi + b\eta + 2t - (b + d)(\xi + \eta - 2t = \frac{(b - d)(\xi + \eta - 2t)}{\xi + \eta - 2t} < 0.$$  

It thus holds that $\bar{C}_1 - C_{10} < 0$ or $\bar{C}_2 - C_{20} < 0$, and $k_1 < 1$ or $k_2 < 1$. Q.E.D.

As to the cases where the solution is either ($\bar{\xi}, \bar{\eta}$) with $0 < \bar{\xi} < \xi$ [case (2-3)] or ($\bar{\xi}, \bar{\eta}$) with $0 < \bar{\eta} < \eta$ [case (3-2)], it has been shown that the paradox should not occur, under the assumption that

$$b \geq 0, d \geq 0, 0 < \xi \leq 5\eta, 0 < \eta \leq 5\xi.$$  

(16)

It can be seen as follows:

Consider case (2-3). From (10) the solution ($\tilde{x}, \tilde{y}$) is ($\bar{\xi} - \bar{\eta}/2 + \eta/2, \eta$), and the conditions to be satisfied are

$$0 \leq 2\bar{\xi} - \bar{\eta} + \bar{\xi} \leq 2\xi \text{ and } \bar{\eta} < \eta.$$  

Thus,

$$\tilde{x} = \bar{\xi} - \bar{\eta}/2 = \frac{1}{2(b + d)}(a - c + 2b\xi + b\eta - t).$$  

(17)

Therefore,

$$\xi - \tilde{x} = \frac{1}{2(b + d)}(a - c + 2d\xi - b\eta + t).$$  

(18)

From the above conditions, by denoting $b + d$ by $B$,

$$2B\tilde{x} = a - c + 2b\xi + b\eta - t \geq 0,$$

$$2B(\xi - \tilde{x}) = c - a + 2d\xi - b\eta + t \geq 0,$$

$$B(\bar{\eta} - \eta) = (c - a)/3 - b\eta - t \geq 0.$$  

(20)

(21)

Then, for case (2-3),

$$(1/\eta)(\bar{C}_2 - C_{20}) = a + b(\bar{\xi} - \tilde{x} + \eta) + t - (c + d\eta).$$

Thus,

$$(2B/\eta)(\bar{C}_2 - C_{20}) = 2B(a - c + t + (b - d)\eta) + b(c - a + t + 2d\xi - b\eta) = (b + d)(a - c) + (3b + 2d)/3 + 2b\xi + 2bd\xi - 2d^2\eta = (b + d)(a - c + 3\eta + 3\xi) - 4a - 2(b + d)^2 + 2bd(\xi - 5\eta).$$

Therefore, from relation (21), if it is assumed $\xi \leq 5\eta$, $\bar{C}_2 - C_{20} < 0$,

which implies no paradox. Similarly for the case (3-2).

For the cases other than the above assumption (16), nothing can be said definitely now.

Note the following: In case (2-2), it holds that

$$(b + d)(\bar{C}_1 - C_{10}) = -(a - c)^2/9 + (b + c + d)\xi + bd(\xi + \eta)\xi + 2(a - c)/3 + b(\eta - \xi) - t^2,$$

$$+(b + d)(\bar{C}_2 - C_{20}) = -(a - c)^2/9 + (b + c + d)\eta + bd(\xi + \eta)\eta + 2(c - a)/3 + d(\eta - \xi) - t^2.$$  

(22)

(23)

(24)

(25)

By noting definitions (11) and (12), eqs. (24) and (25) are expressed as follows:

$$\bar{C}_1 - C_{10} = b\xi / (b + d) - t^2,$$

$$\bar{C}_2 - C_{20} = d\eta / (b + d) - t^2.$$  

(26)

(27)

Therefore, the paradoxical cost degradation occurs when $b\xi / (b + d) < t^2$ and $d\eta / (b + d) < t^2$ hold together.

3.3 The case where the paradoxical cost degradation is worst

Consider the paradoxical situation worst when the minimum value $k_{min}$ among ($k_1, k_2$) is largest (and it is greater than 1). It is natural to call $k_{max}$ the measure of the paradoxical cost degradation. Denote by $a$ the vector $(a, b, c, d, \xi, \eta, t)$ of the parameter values. Denote by $g_i = g_i(a)$ a function of a related to location $i, i = 1, 2$. Assume $b + d > 0$ reflecting Proposition 2.

Lemma 1. If $\sum_i g_i\bar{C}_i$ has the largest value when (and, respectively, only when) the complete symmetry holds, $k_{min}$ has the largest value in (and only in) complete symmetry if $\sum_i g_iC_i$ remains unchanged between the completely symmetrical and asymmetrical cases.

[Proof] Denote the variables in the completely symmetrical case and in an arbitrary asymmetrical case, respectively, with sym and asym. Naturally $k_{min} = k_{min}^{sym}$.

Then from the assumption,

$$k_{min}^{sym} \geq \sum_i g_i^{sym}C_i^{sym} \geq \sum_i g_i^{sym}C_i^{sym} \geq \sum_i g_i^{sym}C_i^{sym}.$$  

Therefore, it follows that $k_{min}^{sym} \geq (\text{and }>) k_{min}^{asym}$: Q.E.D.

From (26) and (27) it follows that

$$\bar{C}_1 - C_{10} + \bar{C}_2 - C_{20} = 2t - (b + d)/2(\bar{x} + \bar{y})(\bar{\xi} - \bar{\eta})^2.$$  

(28)

Call the situation where the values of all the parameters describing each locations are identical in the two locations complete symmetry.
Lemma 2. Consider an arbitrary combination of the values of variables \(a \geq 0\), \(\epsilon \geq 0\), \(b, d, \xi > 0\), \(d\xi > 0\) that satisfy the constraint \(C_{10}/\xi + C_{20}/\xi = R\). (\(R\) is a constant.) Then \(C_1/\xi + C_2/\xi\) has the largest value when complete symmetry holds, i.e., \(a = c, b = d\) and \(\xi = \eta\).

[Proof] It is clear from (28) by noting that, in complete symmetry, \(\tilde{x} = \tilde{y}\). Q.E.D.

Proposition 4. The measure of paradoxical cost degradation, \(k_{\min}\), of the completely symmetric system has the largest value among the systems that have the same value of \(C_{10}/\tilde{x} + C_{20}/\tilde{x}\) with fixed and that are in case (2-2).

[Proof] Follows directly from the two lemmas above. Q.E.D.

Theorem 1. For any system considered here there exist completely symmetrical systems that have measures of paradoxical cost degradation, \(k_{\min}\), not less than the measure of the given system.

[Proof] Obtain the value of \(Z = C_{10}/\tilde{x} + C_{20}/\tilde{x}\) for a system in question. Then obtain a completely symmetric system that has the same value of \(Z\). From Proposition 4, the latter symmetric system has the value of \(k_{\min}\), that is not less than the measure of the former system in question. Denote \(\xi + \eta\) by \(X\). Recall that \(B = b + d\). Note that, in symmetric systems, \(C_{10} = C_{20} = X/2(a + BX/4)\) \((a = c)\) and \(\tilde{x} = \tilde{y} = (BX/4 - t)/B\), and, therefore, the values of \(B, X,\) and \(a\) can be chosen s.t. \(Z = BX(BX/4 + a)/BX/4 - t\). Q.E.D.

Remark. Note that, so far, the worst paradox has been discussed under the assumption (16). The following two subsections, however, give, without assuming (16), some clear examples where complete symmetry is a necessary and sufficient condition for generating the worst measure of paradoxical cost degradation.

3.3.1. The case where \(a = c\). In this case, cases (2-3) and (3-2) cannot exist. This can be seen as follows. Consider the case (2-3). From relation (10), the following two relations have to hold together.

\[0 \leq 2x - y + \eta \leq 2\xi + \eta < \tilde{y}\]

The first inequality of the former relation implies \(a - c - t > -2b\xi - bn\eta\). The second inequality implies \(a - c - t < 2d\xi - bn\eta\).

From definition (12), the latter relation is identical to \((c - a)/3 - bn - t > 0\). From the above two relations, it follows that \((a - c)/3 - t > -b\xi\), which implies \(b > 0\) if \(a = c\). However, this contradicts the latter relation shown above. Thus case (2-3) cannot hold if \(a = c\).

Similarly, it can be shown that case (3-2) can not hold if \(a = c\).

Lemma 3. Consider an arbitrary combination of the values of variables \(a = c > 0\), \(b, d, \xi \geq 0\), \(\eta \geq 0\) that satisfy the constraint:

\[a = c, \xi + \eta = X, (A, B, X; \text{constants}).\] (29)

Then \(C_1 + C_2\) has the largest value if and only if the complete symmetry holds, i.e., \(b = d\) and \(\xi = \eta\).

[Proof] Cases (1-1), (1-2), (1-3), (2-1), (3-1), and (3-3) have \(k_{\min} \leq 1\) as shown in Proposition 3. As noted above, cases (2-3) and (3-2) cannot hold if \(a = c\). Therefore, only case (2-2) remains. Then it follows from (22), (23), (24) and (25) that

\[B(C_1 + C_2) = -2(a - c)^2/9 + (bc + ad\xi + bdX^2) + (b\xi + d\eta)t - 2t^2,\]

\[B(C_2 - C_2 - C_2) = -2(a - c)^2/9 + (c - a)(b\xi - d\eta) - (b\xi - d\eta)^2 + (b\xi + d\eta)t - 2t^2.\]

Therefore the following holds for \(a = c,\)

\[(b + d)(C_1 + C_2 - A) = (b + d)(C_1 - C_{10} + C_2 - C_{20}) = -((b\xi - d\eta)^2 + (b\xi + d\eta)t - 2t^2 = [-B(\xi - \eta) + BX - (b - d)](b - d)/B^2 /4\]

\[+B^2 X^2 - (BX - t)^2)/(4B^2) + 2t(BX/4 - t).\]

(32)

The above is obtained by noting that

\[2(b\xi - d\eta) = (b + d)(\xi - \eta) + (b - d)(\xi + \eta), (33)\]

\[2(b\xi + d\eta) = (b + d)\xi, (b + d)\xi = \xi + \eta, (34)\]

Since \(B = b + d > 0, b > 0\) or \(d > 0\). It is assumed that \(d > 0\). Since \(\tilde{x} = \tilde{y} = 0\), from (11) and \(a = c\), it follows that \(b\xi \geq t\). Then

\[BX = (b + d)(\xi + \eta) \geq (b + d)\xi \geq b\xi \geq t > 0\]

Similar arguments hold if it is assumed that \(b > 0\). Thus \(B^2 X^2 - (BX - t)^2 > 0\).

It is seen from (32) that \(C_1 + C_2\) has the largest value \(2t(BX/4 - t)\) if and only if the complete symmetry holds, i.e., \(b = d\) and \(\xi = \eta\). Q.E.D.

Remark. It should be noted that the paradoxical degradation of performance occurs in complete symmetry only when

\[t < BX/4.\] (35)

Since \(2t(BX/4 - t) = -2(t - BX/8)^2 + B^2 X^2/32\), it follows that the value of \(C_1 + C_2\) in complete symmetry is, furthermore, of the largest value \(A + BX^2/32\) when \(t = BX/8\). Then the largest value of \(k_{\min}\) is \(1 + B^2 X^2/(32A)\).

From the above lemmas, the following proposition follows.

Proposition 5. The measure of paradoxical cost degradation, \(k_{\min}\), of a system has the largest value among the systems that satisfy the constraint (29) with \(t\) fixed, if and only if the system has complete symmetry.

3.3.2. The case where \(b = d\) and \(\xi = \eta\). In this case again, cases (2-3) and (3-2) cannot exist. This can be seen as follows. Consider the case (2-3). From relation (10), the following two relations have to hold together.

0 ≤ 2\tilde{v} + \eta ≤ 2\xi and \eta < \tilde{y}.

The first inequality of the former relation implies \(a - c - t > -2b\xi - bn = -(3/4)BX\).

From definition (12), the latter relation is identical to 
\((c - a)/3 - b\eta - t > 0\) and thus \(c - a - 3t > (3/4)BX\).

From the above two relations, it follows that \(-4t > 0\), which is impossible. Similarly, it can be shown that case (3-2) does not exist if \(a = c\).

Lemma 4. Consider an arbitrary combination of the values of variables \(a\) and \(c\) that satisfy the constraint:
\[
b = d = B/2, \xi(a + b\xi) + \eta(c + d\eta) = A, \xi = \eta = X/2 \quad (A, B, X: \text{constants}). \quad (36)
\]
Then \(\hat{C}_1 + \hat{C}_2\) has the largest value if and only if the complete symmetry holds, i.e., \(a = c\).

[Proof] Cases (1-1), (1-2), (1-3), (2-1), (3-1), and (3-3) have \(k_{min} \leq 1\) as shown in Proposition 3. Cases (2-3) and (3-2) do not exist if \(b = d\) and \(\xi = \eta\) as is noted above. Therefore, only case (2-2) remains. It is thus necessary to consider only case (2-2). Then it follows from (31) that, for \(b = d\) and \(\xi = \eta\),
\[
(b + d)(\hat{C}_1 + \hat{C}_2 - A) = -2(a - c)^2/9 + 2(2B/X - 4 - t) \quad (37)
\]
Therefore, it is seen from (37) that \(\hat{C}_1 + \hat{C}_2\) has the largest value \(2(2B/X - 4 - t)\) if and only if the complete symmetry holds, i.e., \(a = c\). Q.E.D.

Remark. The Remark in the previous subsubsection applies here.

From the lemmas above, the following proposition is obtained.

Proposition 6. The measure of paradoxical cost degradation, \(k_{min}\), of a system has the largest value among the systems that satisfy the constraint (36) with \(t\) fixed, if and only if the system has complete symmetry.

3.3.3. The worst measure

Lemma 5. The worst measure of paradoxical cost degradation for completely symmetrical systems is \(9/8\).

[Proof] As is shown in the Remarks in previous sub-sections, the largest value of \(k_{min}\) is \(1 + BX^2/(32A)\). By noting that \(A = X(a + BX)/4\) \((a = c)\) in complete symmetry, \(A \geq BX^2/4\). Therefore, \(k_{min} \leq 1 + 1/8\). Q.E.D.

Theorem 2. The worst measure of paradoxical cost degradation for systems considered in this paper is \(9/8\).

[Proof] Follows directly from Theorem 1 and Lemma 5. Q.E.D.

Remark. This is in contrast to a KAKH network with non-linear link costs for which the measure of paradoxical cost degradation can be unlimitedly large (Kameda et al., 2000; Kameda and Pourtallier, 2001).

4. CONCLUDING REMARKS

The present paper has examined a model consisting of two locations each with one facility that produces a single kind of commodity. It is counter-intuitive that there exist paradoxical cases where adding transportation capacity between locations may result in cost degradation (i.e., cost increase) to all the locations in the scheme where each location independently makes decisions to optimize its cost of producing the commodity demanded there. In an analytically tractable model, a condition under which such degradation does not occur in the model was obtained. Furthermore, it was shown that the measure of such paradoxical cost degradation to all locations is worst when the model is in complete symmetry under some conditions.

The model examined is rather special, but the results give insight into the problems, and may be expected to hold in more general situations or in different contexts. This suggests directions for further investigation.

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